

The Temporal Resolution of Uncertainty and the Irreversibility Effect

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October 2004

Abstract

We define the irreversibility effect and demonstrate its importance in problems involving investment decisions under uncertainty. We establish several analytical and numerical results that suggest both that the effect holds more widely than generally recognized, and that an existing result (Epstein's Theorem) giving a sufficient condition for determining whether the effect holds can be applied more widely than previously indicated, in particular to problems involving intertemporally nonseparable benefit functions. We further show that a low elasticity of intertemporal substitution will however result in failure of the effect, but that the effect will hold if the value of information increases in the degree of flexibility.

This research received financial support from the National Science Foundation grant No. SES-9818642. The authors gratefully acknowledge comments on an earlier draft of some of the material in this paper by Christian Gollier and participants at the International Conference on Risk and Uncertainty in Environmental and Resource Economics, Wageningen, The Netherlands, and Columbia Earth Institute Environmental Economics Seminar.

1. INTRODUCTION

Environmental impacts of an investment in resource development can be long lasting, or even irreversible. This is a feature of environmental valuation and decision problems that has received a great deal of attention in the literature, based on findings in the natural sciences. For example, there is both scientific and popular concern today about loss of biodiversity, the genetic information that is potentially valuable in medicine, agriculture, and other productive activities. Much of the concern is for endangered species, or the habitats such as tropical moist forests that are subject to more or less irreversible conversion to other uses. But even if species survival is not at issue, biological impacts can be very difficult to reverse over any relevant time span. The clear-cutting of a climax forest species, for example, removes the results of an ecological succession that may represent centuries of natural processes. Regeneration may not lead to the original configuration, as opportunistic species such as hardy grasses come in and preempt the niche otherwise filled by the climax species (Albers and Goldbach 2000).

Irreversibilities have also been identified as a key feature of the problem of how to respond to potential impacts of climate change. Emissions of greenhouse gases, in particular carbon dioxide, accumulate in the atmosphere and decay only slowly. According to one recent calculation, assuming business-as-usual use of fossil fuels over the next several decades, after a thousand years carbon dioxide concentrations will still be well over twice the current level, and nearly three times the pre-industrial level - and will remain elevated for many thousands of years (Schultz and Kasting 1997). There is also some prospect of essentially irreversible catastrophic impact as would result for example from the disintegration of the West Antarctic Ice Sheet and consequent rise in sea level of 15-20 feet. Recent findings suggest that this possibility is more serious, and perhaps closer in time, than economists (and others) have realized ((Kerr 1998) and (de Angelis and Skvarca 2003)).

Irreversibilities are of course not confined to environmental decisions, but occur in a wide variety of economic settings, as the definitive work on investment decisions under uncertainty by Dixit and Pindyck (1994) makes clear.

In the environmental economics literature the analysis of investment decisions under uncertainty and irreversibility was introduced by Arrow and Fisher (1974) and Henry (1974), who show that, for a linear net benefit function or an all-or-nothing choice, it will be optimal to delay or reduce investment, for example in a water resource development project in a natural environment, if future

net benefits are uncertain, investment decisions are irreversible, and there is a possibility of learning about future benefits. Dixit and Pindyck and others establish essentially the same result for the more general investment problem, broadening the treatment to include nonlinear benefit functions and continuous choices, at the same time greatly enriching the analysis with a rigorous treatment of stochastic optimization.

Beginning with the seminal paper by Epstein (1980) on decision-making and the temporal resolution of uncertainty, and including important contributions by Freixas and Laffont (1984), Jones and Ostroy (1984), Hanemann (1989), Kolstad (1996), Ulph and Ulph (1997), and Gollier, Jullien and Treich (2000), another strand of the literature has focused on the question of whether the rather strong and unambiguous results of Arrow and Fisher, Henry, and Dixit and Pindyck, continue to hold in still more general settings in which the benefit function exhibits properties not considered by these authors.

In this paper we take up the discussion of several aspects of this question. The next section provides a definition of the irreversibility effect that is more general than others in the literature. Section 3 is a reconsideration of Epstein's Theorem, leading to the conclusion that it is more widely applicable than commonly understood. Section 4 is an examination of necessary and sufficient conditions for the effect to hold in a continuous-choice, nonlinear setting. Our main conclusion here is that the effect appears to be quite robust, established in part through a novel application of Epstein's Theorem and in part through critical analysis of necessary and sufficient conditions in the literature. Section 5 explores the relationship to risk aversion and intertemporal substitution and Section 6 the relationship to flexibility and the value of information. Section 7 offers some broad conclusions on the status and significance of the irreversibility effect.

2. THE IRREVERSIBILITY EFFECT: A DEFINITION

The issues we want to explore can be represented in a two-period decision problem. In the first period, the decision maker chooses a variable, x_1 ; in the second period, a variable, x_2 . Net benefits in the first period, denoted by $B_1(x_1)$, are deterministic and depend only on x_1 , but net benefits in the second period, denoted by $B_2(x_1, x_2, z_i)$, are stochastic and are a function both of x_1 , x_2 , and also of z , a random variable that reflects the underlying *uncertainty* about the nature of net benefits.¹ We assume that B_1 is concave and twice continuously differentiable in x_1 , and B_2 is

¹For technical reasons, it turns out to be advantageous to assume that z is a discrete random variable.

concave and twice continuously differentiable in x_1 and x_2 . An issue that will become of some importance is whether or not the benefit function is separable in x_1 and x_2 . In the general case where B_2 is a function of x_1 , the benefit function is said to be nonseparable. If, on the other hand, B_2 were only a function of x_2 and z but not of x_1 , then the benefit function would be said to be separable.

In principle, there are constraints on the first- and second-period choices. C_1 denotes the constraint function for x_1 . A crucial issue in the literature is the extent to which the first period choice of x_1 constrains the future choice of x_2 . In general, we will assume that the first period choice does constrain the second period choice, the constraint on the latter being given by $C_2(x_1)$. The constraint on x_2 could take a variety of forms and, in general, it implies a loss of flexibility in the second period decision. A sharp form of the constraint would be $C_2(x_1) = x_1 > x_2$ which implies that x_2 is constrained to be less than x_1 ; we refer to this, and any such constraint on x_2 , as the *irreversibility* constraint.² Note that, by using a non-separable formulation of the second period net benefit function, we already imply that the first period decision will affect the choice confronting the decision maker in the second period. Making the second period constraint function depend on x_1 introduces a separate element of interdependence between the two choices.

Before the second period decision is made, the decisionmaker receives a signal, denoted by y_j , that reveals some information about z . This is the source of *learning*. The amount of information contained in y depends on how closely related z and y are. Let y and y' denote two potential signals where the correlation between y and z is greater than the correlation between y' and z . y is said to be more informative about z and leads to greater learning about the true nature of z than y' . After the signal is received, the decisionmaker updates her prior expectations about z by formulating a posterior distribution denoted by $\pi_{ij} = p(z = z_i/y = y_j)$ and then chooses x_2 for each signal to maximize the expected benefit over the different states. Let q_j denote the prior probability distribution for y .

With this notation, the dynamic optimization problem is:

$$(1) \quad \max_{x_1 \in C_1} \left(B_1(x_1) + \sum_j q_j \max_{x_2 \in C_2(x_1)} \left[\sum_i \pi_{ij} B_2(x_1, x_2, z_i) \right] \right)$$

²The key here is the inequality; depending upon the interpretation of x_1 and x_2 the irreversibility could alternatively be represented by an inequality running in the opposite direction, $C_2(x_1) = x_1 < x_2$

Finally, we assume that a unique solution exists, and lies in the interior of C_1 . Let x_1^* denote the maximum corresponding to the more informative signal y , and x_1^{**} the maximum corresponding to the less informative signal y' .

The conventional definition of the irreversibility effect in the literature is

$$(2) \quad \text{either } x_1^* \geq x_1^{**} \text{ or } x_1^* \leq x_1^{**}$$

If the decision is how much wildlife habitat to keep intact and not convert to farmland, then an increase in the initial choice implies an irreversibility effect. In this case the irreversibility effect holds if $x_1^* \geq x_1^{**}$. On the other hand if the decision is how much of a greenhouse gas to emit when damages due to global warming are uncertain, then a decrease in the initial choice implies an irreversibility effect. In this case the irreversibility effect holds if $x_1^* \leq x_1^{**}$.

An alternative definition, due to Freixas and Laffont (1984), is

$$(3) \quad C_2(x_1^*) \supseteq C_2(x_1^{**}).$$

The irreversibility effect is said to hold if the second period choice set associated with x_1^* is at least as large as the choice set associated with x_1^{**} . Freixas and Laffont also use this relationship to define flexibility.³

We propose a third, more general definition. Define \hat{x}_1 as the value of x_1 that gives maximum decisionmaking flexibility in the future. For example, if x_2 is constrained to be greater than (less than) x_1 , $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$, then $\hat{x}_1 = 0$ ($\hat{x}_1 = 1$). This is because with $x_1 = 0$ ($x_1 = 1$) there is no constraint on the choice of x_2 , and so there is maximum decisionmaking flexibility. In

³There is a second definition of flexibility in the literature due to Jones and Ostroy (1984). Rather than the set of choice variables, Jones and Ostroy define flexibility in terms of the set of second period positions that can be attained from the first period position at a given cost and for a particular state of the world. Let $c(x_1, x_2, z_i)$ denote the cost of moving from x_1 to x_2 given that the state of the world is z_i . Then $G(x_1, z_i, \alpha)$, where

$$G(x_1, z_i, \alpha) \equiv \{x_2 : c(x_1, x_2, z_i) \leq \alpha\},$$

is the set of second period positions attainable from x_1 at a cost that does not exceed α in state s . In general x_1^* is said to be more flexible than x_1^{**} when for all $\alpha \geq 0$ and for all z_i , $G(x_1^*, z_i, \alpha) \supseteq G(x_1^{**}, z_i, \alpha)$. In section 3 we show that the definition of flexibility due to Freixas and Laffont is too restrictive, and can lead to the conclusion that the irreversibility effect is violated when in fact it is not. In the same section we also show that the definition due to Jones and Ostroy proves to be more general.

terms of the second period choice set \hat{x}_1 implies a choice set that consists of all the possible value of the second period choice, x_2 . We will say that an irreversibility effect exists if

$$(4) \quad |x_1^* - \hat{x}_1| \leq |x_1^{**} - \hat{x}_1|,$$

that is, if the optimum corresponding to the more informative signal is at least as close to the point of maximum flexibility as the optimum corresponding to the less informative signal. In some models \hat{x}_1 may be a constant, while in others it may be a function of the model parameters. The virtue of this definition is that it is more widely applicable than either of the alternatives. Note that it encompasses both cases under equation (2) and is independent of the structure of the problem. It is equivalent to $x_1^* \geq x_1^{**}$ in cases where x_2 is constrained to be less than x_1 , $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$. In such cases, $\hat{x}_1 = 1$, and according to our definition, the irreversibility effect holds if $|x_1^* - 1| \leq |x_1^{**} - 1|$. Since x_1 lies between 0 and 1, this simplifies to $x_1^* \geq x_1^{**}$. Alternatively, if x_2 is constrained to be greater than x_1 , $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$, then since $\hat{x}_1 = 0$, our definition simplifies to $x_1^* \leq x_1^{**}$.

In the next section we turn to Epstein's theorem, and after establishing the theorem we discuss two applications. Both applications show that, contrary to a common perception in the literature, Epstein's sufficient condition can be used to establish whether the irreversibility effect holds in models with intertemporally nonseparable benefit functions. In addition, the second application also shows that the conventional definitions of the irreversibility effect based on equations (2) and (3) are too restrictive and that our definition in equation (4) is more general.

3. EPSTEIN'S THEOREM AND ITS APPLICATIONS

Epstein (1980) establishes a sufficient condition under which the initial level of investment in a two-period model with uncertainty and the possibility of future learning is less than the initial level with uncertainty and no or less learning. Using the model in section 2, we can state Epstein's sufficient condition as follows. Let $J(x_1, \xi)$ denote the value function, which is defined as

$$(5) \quad J(x_1, \xi) \equiv \max_{x_2 \in C_2(x_1)} \sum_i \xi_j B_2(x_1, x_2, z_i)$$

where $\xi = [\xi_1, \xi_2, \dots, \xi_j \dots \xi_N]$ and $\xi_j = [\pi_{1j}, \pi_{2j}, \dots, \pi_{ij} \dots, \pi_{Mj}]$ and is a vector of the posterior probability distribution corresponding to the signal y_j . Assume that $J(x_1, \xi)$ is concave and differentiable with respect to x_1 .⁴ The sufficient condition relating x_1^* to x_1^{**} is given in Theorem 1.

Theorem 1. *If $J_{x_1}(x_1^*, \xi)$ is a concave (convex) function of ξ , then $x_1^* \leq (\geq) x_1^{**}$,*

where $J_{x_1}(x_1, \xi)$ is the slope of the value function with respect to its first argument. In words, the sufficient condition states that if the slope of the value function with respect to x_1 is concave (convex) in the posterior probability distribution, then the optimal choice of x_1 associated with the more informative signal is less (more) than the optimal choice associated with the less informative signal.

Epstein's proof is as follows:

Proof. By assumption x_1^* is the unique solution to $\sum q_j J_{x_1}(x_1^*, \pi_j) = 0$ and x_1^{**} is the unique solution to $\sum q'_j J_{x_1}(x_1^{**}, \pi'_j) = 0$, where π_j and π'_j denote the j -th columns of the posterior probability distribution associated with the more and less informative signals. Suppose that $J_{x_1}(x_1^*, \xi)$ is convex in ξ . Since y is more informative than y' , $0 = \sum q_j J_{x_1}(x_1^*, \pi_j) > \sum q'_j J_{x_1}(x_1^*, \pi'_j)$. Therefore $x_1^* \geq x_1^{**}$. Similarly, if $J_{x_1}(x_1^*, \xi)$ is concave in ξ , then $-J_{x_1}(x_1^*, \xi)$ is convex in ξ and it follows that $x_1^* \leq x_1^{**}$.

□

We want to offer three observations regarding this theorem. First, as Epstein clearly states, this is a sufficient condition, not a necessary condition. Second, although the theorem is widely seen as providing a condition for the existence of the irreversibility effect, irreversibility per se does not affect the condition because the constraint that defines the irreversibility, C_2 , plays no specific role in the proof of theorem. Third, because Epstein illustrates the use of his theorem by applying it to some particular models that have an intertemporally separable benefit function, it is sometimes thought that his condition can only be used to investigate the irreversibility effect if there is an intertemporally separable benefit function.⁵ However, Epstein's condition can in fact be used to check whether the irreversibility effect holds when there is an intertemporally nonseparable benefit

⁴This assumption holds if $B_2(x_1, x_2, z)$ is concave in x_1 and x_2 and if for $C_2(x_1) = \{x_2 | f(x_1, x_2) \geq 0\}$, the function f is concave (Epstein 1980).

⁵For example, see (Ulph and Ulph 1997) page 637 and (Gollier et al. 2000) page 233.

function. We show this first in Ulph and Ulph's model of the optimal control of greenhouse gas emissions, and second in Epstein's model of the firm's-demand-for-capital.

3.1. First Application: Non-Separability. To show that Epstein's condition can be applied to a problem characterized by an intertemporally nonseparable benefit function—the control of greenhouse gas emissions, as modeled by Ulph and Ulph (1997)—consider the following dynamic optimization problem:⁶

$$(6) \quad \max_{x_1 \geq 0} \left(B_1(x_1) + \sum_j q_j \max_{x_2 \geq \delta x_1} \sum_i \pi_{ij} (B_{21}(x_2 - \delta x_1) - z_i B_{22}(x_2)) \right)$$

where x_1 is the flow of greenhouse gas emissions in the first period, x_2 is the stock of greenhouse gases in the second period, $1 - \delta$ is the rate of decay of the stock of greenhouse gases, z_i is a random variable, B_1 and B_{21} are concave benefit functions and B_{22} is a convex damage function. Let the j th element of $J(x_1, \xi) = \max_{x_2 \geq \delta x_1} \sum_i \pi_{ij} U(x_2 - \delta x_1) - z_i D(x_2)$. Further assume that the benefit and damage functions are quadratic and let $B_1(x_1) = a_1 x_1 - 0.5 a_2 x_1^2$ and $B_{22}(x_2) = 0.5 a_3 x_2^2$ where $\frac{a_1}{a_2} > x_1$ so that marginal utility is positive.

For the second period, the j th element of the Lagrangian is given by

$$L = \max_{x_2 \geq \delta x_1} \left(B_{21}(x_2 - \delta x_1) - \sum_i \pi_{ij} z_i B_{22}(x_2) + \lambda (x_2 - \delta x_1) \right)$$

where λ is the Lagrangian multiplier. The first order condition for the optimal choice in the second period, x_2^* , is then

$$(7) \quad \frac{\partial B_{21}(x_2 - \delta x_1)}{\partial x_2} - \frac{dB_{22}(x_2)}{dx_2} \xi_j z_i + \lambda \leq 0$$

where $\xi_j = [\pi_{1j}, \pi_{2j}, \dots, \pi_{ij}, \dots, \pi_{Mj}]$. We assume that x_2 is strictly positive so that the inequality in equation (7) holds with an equality. If the inequality constraint on x_2 is not binding then $\lambda = 0$ and the j th element of x_2^* is given by

⁶As stated this problem is a slight generalization of the one by Ulph and Ulph in that it allows for a range of learning levels while Ulph and Ulph allow for either perfect or no learning.

$$x_2^* = \frac{a_1 + a_2\delta x_1}{a_2 + a_3\xi_j z_i}$$

If the constraint binds then $x_2^* = \delta x_1$. By the envelope theorem the j th element of the slope of the value function in x_1 is (and for $\lambda = 0$),

$$J_{x_1}(x_1, \xi) = -a_1\delta + \frac{a_1 a_2 \delta - a_2 a_3 \delta^2 x_1 \xi_j z_i}{a_2 + a_3 \xi_j z_i},$$

The ij th element of the first derivative of the slope of the value function in $\xi = [\xi_1, \xi_2, \dots, \xi_j, \dots, \xi_N]$ is

$$\frac{\partial J_{x_1}(x_1, \xi)}{\partial \xi} = -\frac{(a_2\delta)^2 a_3 x_1 z_i + a_1 a_2 a_3 \delta z_i}{(a_2 + a_3 \xi_{ij} z_i)^2} < 0$$

and that of the second derivative is

$$\frac{\partial^2 J_{x_1}(x_1, \xi)}{\partial \xi^2} = \frac{2(a_2 a_3 \delta z_i)^2 x_1 + 2a_1 a_2 \delta (a_3 z_i)^2}{(a_2 + a_3 \xi_{ij} z_i)^3} > 0$$

This implies that the slope of the value function is convex in ξ^7 and thus $x_1^* \geq x_1^{**}$. Since x_2 is constrained to be greater than δx_1 a lower level of x_1 in the first period implies a greater level of flexibility. For this problem, then, the irreversibility effect is violated. We have however shown that Epstein's condition can be used to establish whether the effect holds in the case of an intertemporally non-separable net benefit function.

That the irreversibility effect does not hold in the case of global warming seems somewhat counter-intuitive, but a recent contribution by Gollier et al. (2000) suggests an explanation. Within the class of models characterized by hyperbolic absolute risk aversion (HARA) preferences, that is, with utility functions

$$(8) \quad B(x) = \frac{\gamma}{1-\gamma} \left[\eta + \frac{x}{\gamma} \right]^{1-\gamma},$$

⁷Note that if $\lambda \neq 0$ then $J_{x_1}(x_1, \xi) = 0$ and $J_{x_1}(x_1, \xi)$ is neither concave nor convex in ξ .

where x is a function of x_1 and x_2 , the coefficient of absolute risk aversion is $\eta + \frac{x}{\gamma}$,⁸ and the slope of the value function is concave (convex) in the random variable if and only if $\gamma < 1$ ($\gamma > 1$ or $\gamma < 0$).⁹

To see how this explains the counter-intuitive result of Ulph and Ulph, consider the coefficient of relative risk aversion associated with the net benefit function in the second period, that is, with $B_2(x_2) = B_{21}(x_2 - \delta x_1) - \xi_j z_i B_{22}(x_2)$. Since

$$\frac{dB_2(x_2)}{dx_2} = a_1 - a_2(x_2 - \delta x_1) - a_3 x_2 \xi_j z_i$$

and

$$\frac{d^2 B_2(x_2)}{dx_2^2} = -a_2 - a_3 \xi_j z_i,$$

the coefficient of relative risk aversion is

$$CRRR = \frac{(a_2 + a_3 \xi_j z_i)x_2}{a_1 + a_2 \delta x_1 - (a_2 + a_3 \xi_j z_i)x_2}$$

At x_2^* $CRRR = \infty$. Following Gollier et al, it is this high coefficient of relative risk aversion that leads to the violation of the irreversibility effect. We explore the relationship between risk aversion and the irreversibility effect further in section 5.¹⁰

3.2. Second Application: More General Definition of Irreversibility Effect. Consider next the following problem faced by a profit maximizing firm:

$$(9) \quad \max_{K \geq 0} \left(-cK + \sum_j q_j \max_{L \geq 0} \left(\sum_i \pi_{ij} p_i F(K, L) - wL \right) \right)$$

⁸Note that if $\eta = 0$ in equation (8), then hyperbolic absolute risk preferences reduce to constant relative risk aversion preferences and γ can be interpreted as the coefficient of relative risk aversion.

⁹This result, in fact, is a generalization of Epstein's in his consumption-savings model. For constant relative risk aversion preferences, Epstein establishes that the third derivative of the value function is concave (convex) if $\alpha < (>)1$ where α is the coefficient of relative risk aversion. Gollier et al. (2000) show that this result can be extended to a more general class of preferences, hyperbolic absolute risk aversion, and that the restrictions on the relevant coefficient of risk aversion are not only sufficient but also necessary to establish the sign of the third derivative of the value function.

¹⁰Note that this result is dependent on the functional forms of the benefit and the damage function. It can be shown that if instead of these functions being quadratic one assumes that the benefit and damage functions display constant relative risk aversion then whether or not the irreversibility effect holds depends on whether the coefficient of risk aversion is greater than or less than one.

where K denotes capital, L denotes labor, c is the cost of capital, w is the wage rate, F is a strictly concave production function and p_i is the unknown output price. The firm determines its demand for capital in the first period and its demand for labor in the second period after it receives some information about output prices. Capital is thus quasi fixed while labor is variable. In the second period the firm can neither invest nor disinvest in capital. Since the first period choice, capital, enters the benefit function in the second period the benefit is said to be intertemporally nonseparable.¹¹ The question is whether Epstein's sufficient condition can be used to establish the irreversibility effect, and whether or not there is an irreversibility effect.

According to Epstein's sufficient condition, whether the irreversibility effect holds depends on the second derivative of the slope of the value function in the random variable. For the following constant elasticity of substitution production function

$$F(K, L) = [aK^{-\beta} + bL^{-\beta}]^{-\frac{1}{\beta}}$$

where $a > 0$, $b > 0$, $\beta > -1$, $\beta \neq 0$, $0 < \mu < 1$ (μ being a measure of returns to scale) and the elasticity of substitution, σ , is equal to $\frac{1}{(1+\beta)}$, Hartman (1976) has established that the third derivative of the value function depends on the relationship between the elasticity of substitution and the returns to scale. Specifically, Hartman has shown that if $\sigma > (<)\frac{1}{(1-\mu)}$ then $J_K(K, p_i)$ is concave (convex) in p_i . This combined with theorem 1 implies that if $\sigma > (<)\frac{1}{(1-\mu)}$ then the demand for capital is lower (higher) when there is a possibility of learning than when there is no possibility of learning. Since the demand for capital does not unambiguously increase or decrease with learning Epstein leads the reader to conclude that the irreversibility effect is violated in this

¹¹Note that if the firm was allowed to invest or disinvest in capital in the second period then the problem faced by the firm would become intertemporally separable. Consider the case where the firm is allowed to disinvest in the capital stock, at a cost, in the second period. The problem described by equation (9) would change to

$$(10) \quad \max_{K_1 \geq 0} \left(-c_1 K_1 + \sum_j q_j \max_{L \geq 0, K_2 \leq K_1} \left(\sum_i \pi_{ij} p_i F(K_2, L) - wL + c_2(K_1 - K_2) \right) \right)$$

where K_1 denotes capital in the first period, K_2 denotes capital in the second period, c_1 is the cost of capital in the first period and c_2 is the cost of capital in the second period. Since there is a cost associated with disinvestment $c_2 > c_1$. Equation (10) can be re-written as

$$\max_{K_1 \geq 0} \left((c_2 - c_1)K_1 + \sum_j q_j \max_{L \geq 0, K_2 \leq K_1} \left(\sum_i \pi_{ij} p_i F(K_2, L) - wL - c_2 K_2 \right) \right)$$

Since K_1 does not affect the benefit function in the second period, the problem is intertemporally separable. A similar case can be made for when the firm is allowed to invest in the second period.

example. Others have interpreted this ambiguous result to mean that Epstein's condition cannot be applied to intertemporally non-separable benefit functions.

We shall argue instead that, even if the slope of the value function is concave for some parameter values and convex for other values, there still may be an irreversibility effect because the flexible value of capital, defined below, also changes with the parameters. This further implies that Epstein's condition can in fact be applied to intertemporally non-separable benefit functions and also argues for a more general definition for the irreversibility effect.

Observe that the firm can neither increase nor decrease its capital stock in the second period. Consequently, one cannot tell *a priori* whether a high or a low demand for capital in the first period constitutes a flexibility-enhancing decision. When σ is high so that capital and labor can be easily substituted then a *lower* capital stock today may very well give the decision maker greater flexibility tomorrow. If it turns out that the decision maker has underestimated his or her production needs then he or she can compensate for the low stock of capital by hiring more labor. On the other hand, if σ is low so that capital and labor cannot be substituted, a *higher* capital stock today may maintain greater flexibility tomorrow. If so, when $\sigma > (<) \frac{1}{(1-\mu)}$, a decrease (increase) in the demand for capital when there is a possibility of learning constitutes an irreversibility effect. We show that this is in fact the case, and that the model does give rise to an irreversibility effect, first by defining what is meant by flexibility in this context and then by showing that the level of capital that gives the greatest amount of flexibility is lower (higher) when $\sigma > (<) \frac{1}{1-\mu}$.

If we attempt to apply the Freixas and Laffont definition of flexibility (see section 2) to Epstein's model, then since capital can neither be increased nor decreased in the second period, the set $C_2(x_1)$ is empty, where x_1 and x_2 are the levels of capital chosen in the first and the second period respectively.¹² Defining x_2 as the level of labor chosen in the second period does not help to determine the level of capital that gives more or less flexibility in the second period as the first period's choice of capital in no way restricts the choice of labor in the second period.

By the definition of flexibility put forward by Jones and Ostroy (1984) (see footnote 3), so long as x_1 and x_2 are defined in terms of the choice variables (capital or labor), the set $G(x_1, z_i, \alpha)$ is also empty. This then brings us to the question, with respect to what variables should flexibility be measured? So far we have measured flexibility in terms of the choice variables, so flexibility is

¹²Note that the set C cannot be used to define the irreversibility effect either in this problem.

measured in terms of the choices of capital or labor in the second period that are feasible given the choice of capital in the first period. However, one could instead measure flexibility in terms of the level of output that can be attained in the second period given the choice of capital in the first. After all, the firm cares about the level of capital, or any other input, only in so far as it allows the firm to produce output in the second period. With this alternative measure of flexibility, if we define x_1 as the level of capital chosen in the first period, x_2 as the level of output attained in the second period, z_i as the price of output in the second period and α as the wage rate, then $G(x_1, z_i, \alpha)$ can be defined as the set of outputs that can be attained for given levels of capital and labor and for a particular price of output.

The question is how does one estimate the set $G(x_1, z_i, \alpha)$? One possibility is to define the set in terms of the *range of output*¹³ that can be attained for a given level of capital. If the firm learns that the price of output is likely to be high tomorrow then the firm would like to produce a high output and if it learns that the price of output is likely to be low then it would like to produce a low output. Flexibility for the firm manifests itself in terms of the range of output that the firm can produce. With this definition a more flexible level of capital is one that enables the firm to produce a greater range of output in the second period.

We now show that the most flexible level of capital changes with a change in the parameters, and further, that the level of capital that gives the greatest flexibility is lower (higher) when $\sigma > (<) \frac{1}{1-\mu}$.

Proposition 1. *If $\sigma > (<) \frac{1}{1-\mu}$ then $\hat{K} = \underline{K}(\overline{K})$.*

where \hat{K} is the level of capital that implies the greatest amount of flexibility, \underline{K} is the minimum capital stock and \overline{K} is the maximum capital stock.¹⁴

Proof. Let $\bar{y}(K)$ denote the range of output that can be achieved for a given level of capital and let $\gamma = \frac{-\mu}{\beta}$. Note that when $\sigma > (<) \frac{1}{1-\mu}$, $\gamma < (>) 1$ since $\sigma = \frac{1}{1+\beta} > (<) \frac{1}{1-\mu}$ implies that $\mu < (>) -\beta$.

$$\bar{y}(K) = (aK^{-\beta} + b\overline{L}^{-\beta})^\gamma - (aK^{-\beta} + b\underline{L}^{-\beta})^\gamma$$

¹³Note that this is consistent with Hirshleifer and Riley (1992) who point out that flexibility is different from the range of actions which in our example would mean the range of capital or labor. We instead equate flexibility to the range of outputs.

¹⁴If $\sigma = 1$ so that the production function is a Cobb-Douglas then one can show that a higher level of capital gives a greater range of output. If $\sigma = 0$ so that the production function is a Leontief then it is difficult to determine what level of capital gives greater flexibility tomorrow.

where \underline{L} is the minimum labor and \bar{L} is the maximum labor. The derivative of the range of output with respect to the capital stock is given by

$$\frac{\partial \bar{y}}{\partial K} = -a\gamma\beta K^{-(\beta+1)} \left((aK^{-\beta} + b\bar{L}^{-\beta})^{\gamma-1} - (aK^{-\beta} + b\underline{L}^{-\beta})^{\gamma-1} \right)$$

When $\gamma < (>)1$, $\frac{\partial \bar{y}}{\partial K} < (>)0$. This in turn implies that when $\gamma < (>)1$ then the level of capital that gives the maximum range of output, \hat{K} , is equal to the minimum (maximum) stock of capital. \square

Thus we have shown that the irreversibility effect holds, and Epstein's condition can be applied to intertemporally non-separable benefit functions. Moreover, we have shown that our definition of the irreversibility effect is more general than the conventional definitions.

4. NECESSARY VERSUS SUFFICIENT CONDITIONS

Epstein's theorem, as we have seen, establishes a *sufficient* condition for the irreversibility effect. This means that the effect may still occur even if the condition does not hold. The subsequent literature has developed a number of additional sufficiency conditions, and at least a couple of claimed necessary conditions. The alternative sufficient conditions expand the scope of the effect, but it is our view that the case has not yet been made for the more restrictive and more powerful necessary conditions. In this section we consider the proposed conditions.

4.1. Freixas and Laffont. The quest for a necessary condition was begun by Freixas and Laffont (1984), who develop a necessary and sufficient condition for the irreversibility effect to hold for intertemporally separable net benefit functions. They consider the dynamic optimization problem:

$$(11) \quad \max_{x_1} \left(B_1(x_1) + \sum_j q_j \max_{x_2 \leq x_1} \sum_i \pi_{ij} B_2(x_2, z_i) \right)$$

where x_1 is the choice variable in the first period, x_2 is the choice variable in the second period, z is a random variable, q_j is the prior probability on the signal and π_{ij} is the posterior probability distribution. Let x_1^* be the optimal choice of x_1 under the more informative signal, x_1^{**} the optimal choice under the less informative signal and $J(x_1, \xi)$ be the value function. Theorem 2 specifies the

condition developed by Freixas and Laffont for the irreversibility effect to hold given that a unique solution exists.

Theorem 2. $x_1^* \geq x_1^{**}$ if $B_1(x_1) + J(x_1, \xi)$ is quasi-concave.

The theorem states that the irreversibility effect holds if the value function is quasi-concave or that quasi-concavity is a sufficient condition for the irreversibility effect to hold when the benefit function is intertemporally separable. Freixas and Laffont establish sufficiency analytically and then develop a numerical example to show that the sufficient condition is also necessary. Their numerical example establishes that if quasi-concavity is violated then, in fact, the irreversibility effect is violated. The irreversibility effect (that is, $x_1^* \geq x_1^{**}$ given that $x_2 \leq x_1$) is also shown to be equivalent to $J(x_1, \xi') - J(x_1, \xi)$ being locally increasing in x_1^{**} where ξ' is the more informative signal, ξ is the less informative signal, and $J(x_1, \xi') - J(x_1, \xi)$ is the value of information.

It seems to us that quasi-concavity is only sufficient and not necessary for the irreversibility effect to hold in intertemporally separable functions. By slightly modifying Freixas and Laffont's numerical example, we show that the irreversibility effect can hold even when the value function is not quasi-concave; quasi-concavity is therefore sufficient, not necessary. Having said this, we note that quasi-concavity is a very weak condition. Most if not all empirically important benefit functions will exhibit this property.

In Freixas and Laffont's numerical example the random variable z is assumed to take two possible values, z_1 and z_2 each with probability 0.5. Furthermore, there are two levels of learning, perfect or none at all. The functional forms of the benefit functions are,

$$B_1(x_1, 0 \leq x_1 \leq 2.5\pi) = \pi$$

$$B_1(x_1, x_1 \geq 2.5\pi) = -1.25(x_1 - 2.5\pi) + \pi$$

$$B_2(x_2, z_1) = 2x_2$$

$$B_2(x_2, z_2) = -\cos x_2 + 1$$

With these benefit functions quasi-concavity is violated, $x_1^* \leq x_1^{**}$ and $J(x_1, \xi') - J(x_1, \xi)$, the value of information, is not increasing in x_1 . These results are shown in Figure 1 where the choice variable in the first period, x_1 , is drawn on the x-axis and the value of information on the y-axis.

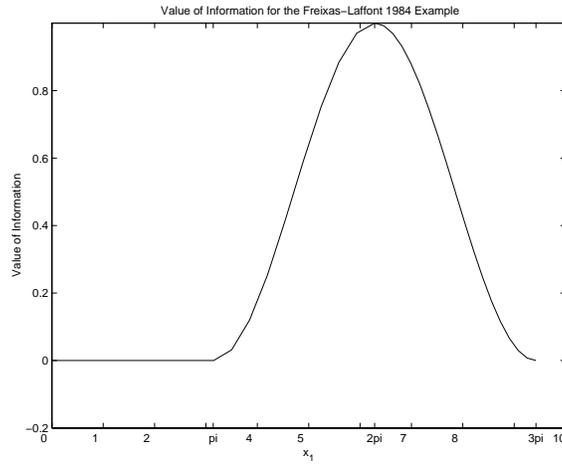


FIGURE 1. Value of Information for the Example in Freixas-Laffont 1984

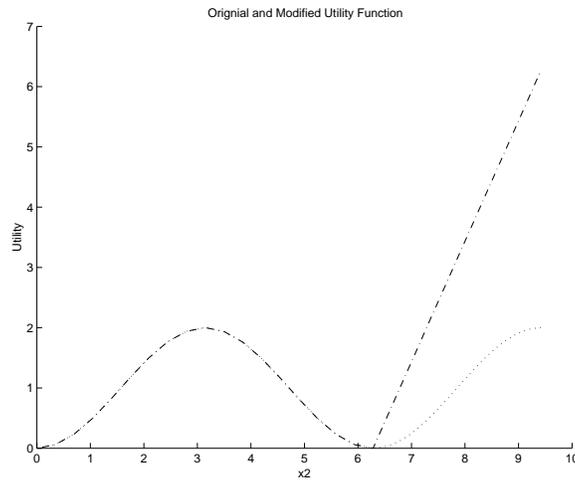


FIGURE 2. Original and Modified Utility Functions

Note that in the range $[2\pi, 3\pi]$ the value of information decreases in x_1 . So long as the optima lie in this range the irreversibility effect is violated.

Now consider a slight modification where $B_1(x_1)$ and $B_2(x_2, z_1)$ remain unchanged and $B_2(x_2, z_2)$ is given by

$$B_2(x_2, z_2, 0 \leq x_2 \leq 2\pi) = -\cos x_2 + 1$$

$$B_2(x_2, z_2, x_2 \geq 2\pi) = x_2 - 2\pi$$

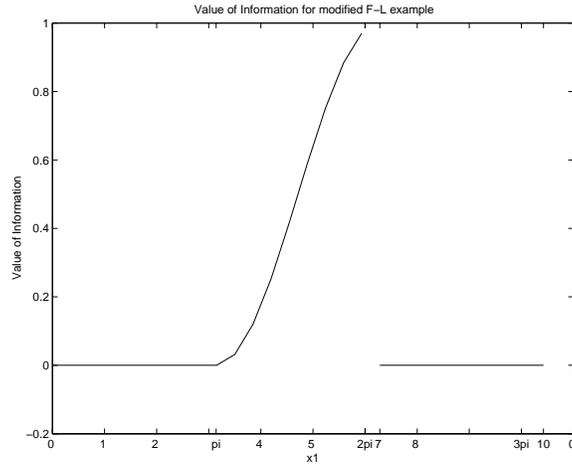


FIGURE 3. Value of Information for the Modified Example in Freixas-Laffont 1984

The original and modified benefit functions are illustrated in Figure 2. Both functions are identical in the range $[0, 2\pi]$ and thereafter the original function is represented by the dotted line and the modified function by the broken line.

With the modified value function although quasi-concavity is still violated, $x_1^* \geq x_1^{**}$ (strictly greater if the optimal lies between π and 2π and equal otherwise) and the value of information is no longer decreasing over a finite interval of x_1 . These results are shown in Figure 3. Note that with the modified benefit function the value of information no longer decreases in the range $[2\pi, 3\pi]$. The irreversibility effect holds though quasi-concavity is violated. Consequently, quasi-concavity is not necessary, merely sufficient, for the irreversibility effect to hold in the class of intertemporally separable benefit functions.

4.2. **Gollier et al.** Another set of sufficient conditions are developed by Gollier et al. (2000) while relating primitives of an economic model to Epstein's otherwise difficult to interpret conditions. As they stand it is not clear what type of model gives rise to a concave or convex slope of the value function and thus qualifies for application of Epstein's Theorem. Gollier et al. (2000) provide necessary and sufficient conditions for two classes of models under which the second derivative of the slope of the value function can be signed. As noted previously, within the class of models characterized by hyperbolic absolute risk aversion (HARA) preferences, the slope of the value function is concave (convex) in the random variable depending on the coefficient of risk aversion. Also, in models with small risks or in which the random variable has a two-atom support, the slope

of the value function is concave (convex) if and only if absolute prudence is larger (smaller) than twice the absolute aversion to risk.

Gollier et al relate these results to a discussion of necessary versus sufficient conditions. They state that their conditions are necessary as well as sufficient to determine whether the initial level of the decision variable with learning is greater or less than the initial level with no or less learning (i.e., whether $x_1^* \lesseqgtr x_1^{**}$), not just necessary and sufficient to sign the third derivative of the value function. They first derive the necessary and sufficient conditions to sign the third derivative of the value function, or the second derivative of the slope, and then on the basis of Epstein's theorem they state that the sign of the third derivative is necessary and sufficient to determine whether $x_1^* \leq x_1^{**}$. Epstein's theorem demonstrates that the sign of the third derivative is a sufficient condition for determining whether or not the initial decision with more learning is less than the initial decision with less learning. Therefore, it appears that the Gollier et al conditions are only sufficient to sign the relationship between first period decisions and learning.

Having considered the effect of learning on first period decisions in the absence of an explicit irreversibility constraint, Gollier et. al introduce an irreversibility constraint and reconsider the effect of learning. They show that the irreversibility effect holds if (but not only if) either prudence is larger than twice absolute risk aversion and the utility function is HARA or prudence is larger than twice absolute risk aversion and the risk is binary or small.

4.3. Ulph and Ulph; Kolstad. Another set of sufficient conditions have been developed by Ulph and Ulph (1997) and Kolstad (1996). Ulph and Ulph develop a new sufficient condition that establishes the irreversibility effect for intertemporally nonseparable net benefit functions with multiplicative uncertainty. In terms of the model described in section 2, multiplicative uncertainty implies that $B_2(x_1, x_2, z_i) = z_i B_2(x_1, x_2)$. For these models, Ulph and Ulph's sufficient condition states that, if the irreversibility constraint bites when there is no possibility of learning, then the irreversibility effect must hold. In terms of our canonical model, let x_2^{**} and x_1^{**} denote the optimal decisions in the absence of learning. If $C_2(x_1) = x_1 \leq x_2$, then $x_2^{**} = x_1^{**}$ implies that the irreversibility constraint bites in the absence of learning. So long as this condition is met, the irreversibility effect is said to hold. Since the condition cannot be applied to models in which the irreversibility constraint does not bite in the absence of learning (i.e., when $x_2^{**} \neq x_1^{**}$) or for

different specifications of uncertainty, it may be considered less general Epstein's, given that the latter can also be applied in cases of intertemporal nonseparability, as we have shown.

In the same vein Kolstad shows that the irreversibility effect holds in models with *effective* irreversibility—that is, in models in which the irreversibility constraint bites. Kolstad builds on an example developed by Freixas and Laffont to establish his sufficient condition:

$$\text{sign}(x_1^* - x_1^{**}) = \text{sign}(\Delta(y) - \Delta(y'))$$

where

$$\Delta(y) = \frac{\partial f_1}{\partial x_1} \sum_{y \in A(y)} \sum_{\pi_{ij}} \frac{\partial B_2}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \sum_{y \in B(y)} \sum_{\pi_{ij}} \frac{\partial B_2}{\partial x_2},$$

$C_2(x_1) = f_1(x_1) \leq x_2 \leq f_2(x_1)$ (so that the initial decision imposes a lower and upper bound on the choice variable in the second period), $A(y) = \{y_j | x_2(x_1, y_j) = f_1(x_1)\}$, a set of signals that end up resulting in an action at the lower bound of the constraint, and finally $B(y) = \{y_j | x_2(x_1, y_j) = f_2(x_1)\}$, a set of signals for which the optimum choice in the second period is at the upper bound of the constraint. Thus, it is necessary to sign $\Delta(y) - \Delta(y')$ in order to establish the irreversibility effect. Though not strict intertemporal separability, Kolstad's sufficient condition requires that $\frac{\partial B_2(x_1, x_2, z_i)}{\partial x_1}$ be independent of x_2 . and may therefore be considered less general than Epstein's.

In summary, to date a number of sufficient conditions have been established for the irreversibility effect, but not necessary conditions, which would be both more restrictive and more powerful. Put differently, the irreversibility effect may hold more widely than previously believed, since it may hold even when the sufficient conditions are violated. The earliest sufficiency result, due to Epstein, we have shown to be more general in the sense of being more widely applicable—to intertemporally nonseparable benefit functions—than is often realized. However, a shortcoming of this condition is that it is not readily related to the primitives of an economic model. This is remedied by Gollier et al, who derive necessary and sufficient conditions on the primitives to sign the second derivative of the slope of the value function, and therefore to apply Epstein's sufficiency theorem. The theorem of Freixas and Laffont is also important in this regard, showing that the irreversibility effect will hold if (and they would argue only if) an intertemporally separable benefit function is quasi-concave, a condition one would normally expect to be satisfied.

Another question unresolved in the existing literature is whether risk aversion can be separated from the irreversibility effect. We examine this issue in the context of the consumption and savings problem discussed by Epstein. In this example an individual allocates an initial amount of wealth between consumption and savings over three periods. Investment in the first period yields a fixed return while investment in the second period yields a random return. Some information is gained about the random rate of return at the beginning of the second period.

$$(12) \quad \max_{0 \leq x_1 \leq w} B_1(w - x_1) + \beta \sum_j q_j \max_{0 \leq x_2 \leq r x_1} \left(B_2(r x_1 - x_2) + \beta \sum_i \pi_{ij} B_3(x_2 z_i) \right)$$

where x_1 and x_2 denote savings in periods 1 and 2 respectively, w is the initial wealth, r is the sure gross rate of return to the first period savings, β is the discount factor, z_i is the random gross return to second period savings, B_1 is the utility function in the first period, B_2 the utility function in the second period and B_3 the utility function in the third period.

With the following constant relative risk aversion utility function,

$$(13) \quad B(c) = \begin{cases} \frac{c^{1-\alpha}}{1-\alpha} & \text{if } \alpha \neq 1, \\ \log(c) & \text{if } \alpha = 1, \end{cases}$$

where α is the coefficient of relative risk aversion, Epstein has established that the effect of learning on the optimal level of savings in the first period depends on the elasticity of intertemporal substitution, that is, on $\sigma = \frac{1}{\alpha}$. When $\sigma > 1$ the slope of the value function is convex and the possibility of learning about the future rate of return leads to an increase in savings in the first period and when $\sigma < 1$ the possibility of learning leads to a decrease in the level of first period savings.

Since savings in the second period are constrained to be no greater than the gross rate of return times savings in the first period and since the level of savings does not unambiguously increase with learning, this is evidence that the irreversibility effect is violated in this problem. Specifically, the irreversibility effect is violated when $\sigma < 1$, that is, when benefits are intertemporally non-substitutable or the coefficient of risk aversion is large. This can be interpreted to imply that risk aversion cannot be separated from the irreversibility effect. However, with constant relative

risk aversion preferences, as with all Von-Neumann-Morgenstern preferences, the coefficient of risk aversion is constrained to be the reciprocal of the coefficient of intertemporal substitution, and therefore one cannot tell whether the violation of the irreversibility effect is being driven by high risk aversion or low intertemporal substitution. We therefore consider an example of generalized isoelastic preferences, for which the coefficient of relative risk aversion is not constrained to be the reciprocal of the elasticity of intertemporal substitution. This allows us to separate the effect of risk aversion from that of intertemporal substitution, and to determine whether the violation of the irreversibility effect is being driven by the lack of intertemporal substitutability or by high risk aversion. This shows, therefore, whether the irreversibility effect can be separated from risk aversion.

Consider the generalized isoelastic preferences:

$$J_t = B(c_t, E_t J_{t+1})$$

$$= \frac{\left((1 - \beta)c_t^{1-\rho} + \beta[1 + (1 - \beta)(1 - \alpha)E_t J_{t+1}]^{\frac{1-\rho}{1-\alpha}} \right)^{\frac{1-\alpha}{1-\rho}} - 1}{(1 - \beta)(1 - \alpha)}$$

where $\beta \in (0, 1)$, $\alpha > 0$ and is the coefficient of relative risk aversion and $\frac{1}{\rho} = \sigma$ is the elasticity of intertemporal substitution. Note that σ is no longer constrained to be equal to $\frac{1}{\alpha}$. With isoelastic preferences it is difficult to establish the relationship between the convexity or concavity of the slope of the value function and α and ρ analytically. Consequently we use numerical simulation to separate the effects of risk aversion and intertemporal substitution. We compare optimal savings in the first period with perfect learning and with no learning for a wide range of parameter values for α and ρ . Simulations give the results in Table 1.

Table 1: Experiments with Generalized Isoelastic Preferences

	$\alpha < 1$	$\alpha > 1$
$\sigma < 1$	$x_1^* \underset{\geq}{\leq} x_1^{**}$	$x_1^* \underset{\geq}{\leq} x_1^{**}$
$\sigma > 1$	$x_1^* \geq x_1^{**}$	$x_1^* \geq x_1^{**}$

When $\sigma < 1$ it is feasible for $x_1^* < x_1^{**}$, that is, for savings to decrease with learning for both $\alpha < 1$ and $\alpha > 1$. However, when $\sigma > 1$ x_1^* is always at least as large as x_1^{**} irrespective of the coefficient

of relative risk aversion. This implies that though the irreversibility effect is violated even with non-expected utility preferences, the violation is caused by a low elasticity of intertemporal substitution and not by a high coefficient of relative risk aversion, supporting an assertion by Epstein that the violation of the irreversibility effect can be attributed to intertemporal substitution rather than to risk aversion (see footnote 13 on page 278 in Epstein (1980)).

6. THE IRREVERSIBILITY EFFECT AND THE VALUE OF INFORMATION

Finally, we consider another important relationship touched on in the literature, between the irreversibility effect and the value of information. As shown by Hanemann (1989), with a continuum of development levels quasi-option value is no longer equivalent to the conditional value of perfect information—conditional on there being no investment initially.¹⁵ However, there is a relationship between the irreversibility effect and the unconditional value of perfect information. If the unconditional value of perfect information increases in the degree of flexibility then the irreversibility effect holds. We establish this relationship numerically through examples previously discussed in the paper.

It turns out that in three, out of the four, examples discussed by Epstein where the irreversibility effect holds (the timing of orders for capital, highways and farms, and the firm's demand for capital) the unconditional value of perfect information is positively correlated with the degree of flexibility. On the other hand, in examples where the irreversibility effect is violated (Epstein's consumption-savings problem and Freixas and Laffont's numerical example) the unconditional value of perfect information decreases even though the level of flexibility increases.

Let the unconditional value of perfect information be defined as $J(x_1, 1) - J(x_1, 0)$ where $J(x_1, 1)$ denotes the value function under perfect learning and $J(x_1, 0)$ denotes the value function under no learning, both evaluated at some level of initial investment x_1 . Note that the value function itself is defined by equation (5). Now consider Epstein's firm's-demand-for-capital example. We have shown that when capital and labor are highly substitutable a low demand for capital in the first period leads to greater flexibility and when capital and labor are not easily substitutable then a high demand for capital implies greater flexibility. Furthermore, as shown in Figures 4 and 5 the unconditional value of perfect information increases in the level of flexibility—the value of information increases in the level of capital with low substitutability and decreases in the level of

¹⁵For a discussion of the relationship between the irreversibility effect and quasi-option value see Hanemann (1989).

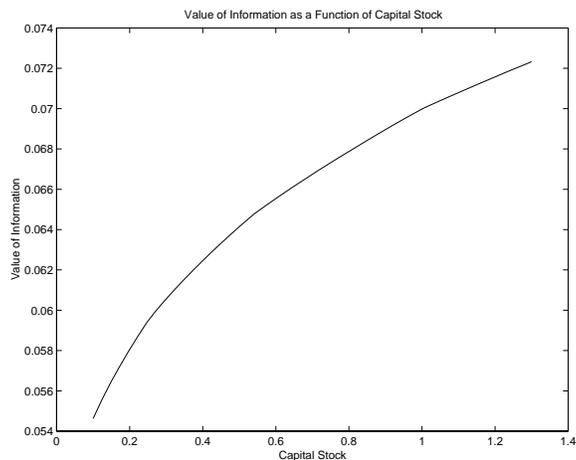


FIGURE 4. Value of Information with Low Substitutability

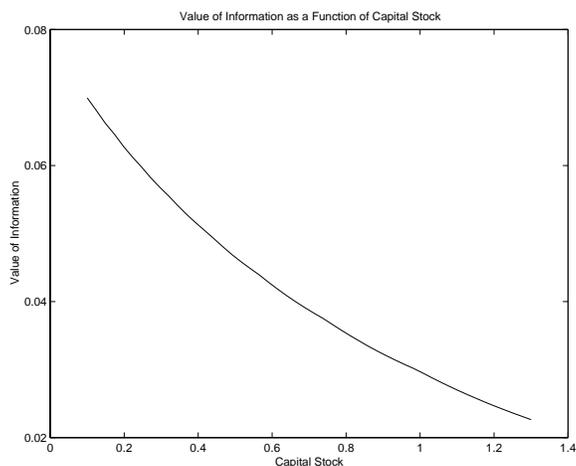


FIGURE 5. Value of Information with High Substitutability

capital with high substitutability. Consequently, for the firm's-demand-for-capital example, if the unconditional value of perfect information increases in the level of flexibility then the irreversibility effect holds.

Now consider Espstein's consumption-savings example. We have shown that flexibility increases in the level of initial savings irrespective of whether the elasticity of intertemporal substitution is high or low. Figure 6 shows that the value of information increases in the level of flexibility, that is in the initial level of savings, when the elasticity of intertemporal substitution is high while Figure 7 shows that the value of information decreases in the level of flexibility when the elasticity of intertemporal substitution is low. In the former case the irreversibility effect holds; in the latter it is violated.

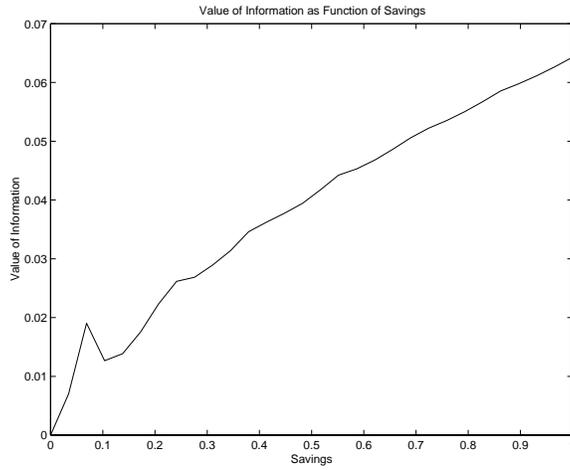


FIGURE 6. Value of Information with High Elasticity of Substitution

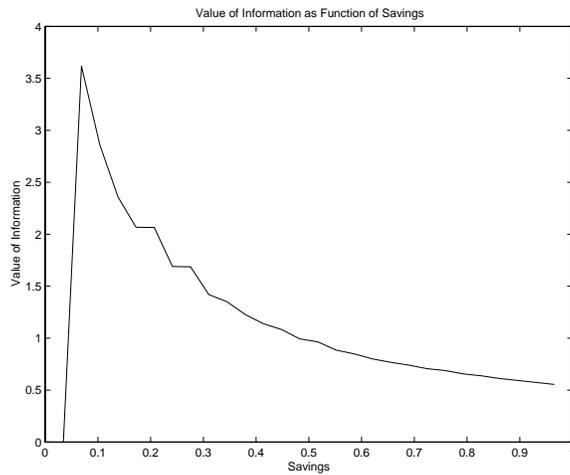


FIGURE 7. Value of Information with Low Elasticity of Substitution

Similarly, in the example discussed by Freixas and Laffont, the irreversibility effect is violated when the value of information does not increase in the level of flexibility. This is illustrated in Figure 1. In the figure, and corresponding numerical example, even though flexibility increases in x_1 , the value of information decreases for $x_1 \in [2\pi, 3\pi]$. When the example is modified to restore the irreversibility effect the value of information becomes a monotonic function of the degree of flexibility, as illustrated in Figure 3.

How can an increase in flexibility be associated with a decrease in the value of information? Over the interval $(2\pi, 3\pi)$ where the value of information decreases in x_1 , an increase in flexibility increases the value function with no learning more than it increases the value function with learning.

This has the effect of decreasing the value of information in x_1 . To see this consider the case where x_1 lies in the interval $[\pi, 3\pi]$.

$$\begin{aligned}
\text{Value of Information} &= \underbrace{\sum_i r_i \max_{x_2 \leq x_1} B_2(x_2, z_i)}_{J(x_1, 1)} - \underbrace{\max_{x_2 \leq x_1} \sum_i r_i B_2(x_2, z_i)}_{J(x_1, 0)} \\
&= 0.5 \max_{x_2 \leq x_1} 2x_2 + 0.5 \max_{x_2 \leq x_1} (1 - \cos x_2) - \max_{x_2 \leq x_1} (x_2 + 0.5 - 0.5 \cos x_2) \\
&= x_1 + 0.5 - 0.5 \cos \pi - x_1 - 0.5 + 0.5 \cos x_1 \\
&= 0.5(\cos x_1 - \cos \pi) \\
&= 0.5(\cos x_1 + 1)
\end{aligned}$$

where r_i is the probability of state z_i occurring. For values of x_1 between π and 2π , $\cos x_1$ increases in x_1 and thus the value of information increases in x_1 over this interval. On the other hand, when x_1 lies between 2π and 3π then $\cos x_1$ decreases in x_1 and so does the value of information. Note that $J(x_1, 1) = x_1$ and

$$\frac{dJ(x_1, 1)}{dx_1} = 1$$

when $x_1 \in [\pi, 3\pi]$. These imply that the value function with learning increase with flexibility (which in turn amounts to an increase in x_1) over the interval $x_1 \in [\pi, 3\pi]$. Also $J(x_1, 0) = 0.5 + x_1 - 0.5 \cos x_1$ and

$$\frac{dJ(x_1, 0)}{dx_1} = 1 + 0.5 \sin x_1.$$

Since $\sin x_1 \in [-1, 1]$ for $x_1 \in [\pi, 3\pi]$, an increase in flexibility also increases the value function with no learning. However,

$$\frac{dJ(x_1, 0)}{dx_1} = \begin{cases} < 1 & \text{for } x_1 \in [\pi, 2\pi], \\ > 1 & \text{for } x_1 \in [2\pi, 3\pi]. \end{cases}$$

Consequently, an increase in flexibility increases the value function with learning more than the value function without learning over the interval $x_1 \in [\pi, 2\pi]$ thereby increasing the value of

information over this interval. However, for $x_1 \in [2\pi, 3\pi]$ an increase in flexibility increases the value function without learning more than the value function with learning, causing the value of information to decrease over this interval.

An interesting question is whether the violation of the irreversibility effect in this example can be attributed to intertemporal non-substitutability as in the consumption-savings example. Unfortunately, it is not possible to define the elasticity of intertemporal substitution for the numerical example because the benefit functions are either linear, in which case the elasticity is undefined, or not quasi-concave, in which case the elasticity is negative. Thus it is not possible to determine whether it is in fact a low elasticity of substitution that is driving the violation of the irreversibility effect.

7. CONCLUDING REMARKS

We have defined the irreversibility effect and indicated its relevance to environmental and other problems involving decisions under uncertainty, and established a number of analytical and numerical results. Our most sweeping conclusion is that it seems to hold more widely than has perhaps previously been recognized. We provide a critical review of conditions established in the literature for the effect to hold. From our review, it appears that these conditions are sufficient, but not necessary. The effect may hold though one or more are violated.

An interesting interpretive result is that Epstein's condition, the original contribution to this literature, and Theorem 1 in our paper, can in fact be applied more widely, in particular to intertemporally nonseparable benefit functions, than previously indicated. Of course this tells us nothing about whether the effect holds in a particular case. We show however using a new and more general definition of irreversibility that it does hold in Epstein's model of the firm's demand for capital, characterized by an intertemporally nonseparable benefit function. By the same token, we use the condition to prove that the irreversibility effect does not hold in another application to an intertemporally nonseparable benefit function, a model of the optimal control of greenhouse gas emissions.

The irreversibility effect is related to other concepts in the literature on decisions under uncertainty. We show with the aid of a numerical simulation involving generalized isoelastic preferences, for which the coefficient of relative risk aversion is not constrained to be the reciprocal of the elasticity of intertemporal substitution, that violation of the irreversibility effect in Epstein's

consumption-savings model is driven by a low elasticity of intertemporal substitution and not by a high coefficient of relative risk aversion.

Numerical analysis of several different models also demonstrates an important relationship between the irreversibility effect and the value of information. If the value of information increases in the degree of flexibility then the irreversibility effect holds. It seems obvious that the greater the flexibility in a decision environment, the more valuable information bearing on the decision will be, and indeed this will generally be the case. Since the value of information is however given by the difference in the value function (in a dynamic programming problem) with and without learning, we show that where the irreversibility effect is violated an increase in flexibility increases the value function with learning by less than the value function without learning, thereby decreasing the value of information.

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